Four-Index Energy-Momentum Tensors for Gravitation and Matter*

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Abstract

The 4-index energy-momentum tensors for gravitation and matter are analyzed on the basis of new equations for the gravitational field with the Riemann tensor. Some properties of the such defined gravitational energy are discussed.

1   Introduction

In general relativity a true and covariant characteristics of the gravitational field is the Riemann curvature tensor R_{iklm}, but the field equations contain only the Ricci tensor, vanishing in the vacuum. The Weyl tensor C_{iklm} which is a nonvanishing in the vacuum pure 4-index part of R_{iklm}, disappear at 2-index contraction [1]. This fact was leading to the problems with the definition of the energy-momentum for the gravitational field.

In the paper [1] a new generalized version of the Einstein equations with the Riemann tensor has been formulated. It has been shown that the 4-index energy-momentum tensors for gravitation and matter can be constructed. In the present paper the structure of new gravitational equations and properties of 4-index energy-momentum tensors will be discussed.

2   Four-index equations for gravitation

We started from the standard Einstein-Gilbert action for the gravitational field:

$$S \;=\; \frac{1}{2}\int d\Omega\sqrt{-g}\left(-\frac{1}{\kappa}R + L\right) = \tag{1}$$

$$=\; -\frac{1}{2}\int d\Omega\sqrt{-g}\left[\frac{1}{2\kappa}(g^{il}g^{km} - g^{im}g^{kl})R_{iklm} - L\right], \tag{2}$$

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where $\kappa = 8\pi k/c^4$. We perform the variational procedure so that R_{iklm} preserves its 4-index form. The result of the such variation is [1]:

$$\delta S_g = -\frac{1}{2} \int d\Omega \sqrt{-g} \delta g^{km} g^{il} [G_{iklm} - T_{iklm}] = 0. \tag{3}$$

Here new 4-index tensors are defined as $(d = 4)$:

$$G_{iklm} = \frac{1}{\kappa} \left[R_{iklm} - \frac{1}{6}(g_{il}g_{km} - g_{im}g_{kl})R \right], \tag{4}$$

$$T_{iklm} = V_{iklm} + \frac{1}{2}(g_{km}T_{il} - g_{kl}T_{im} + g_{il}T_{km} - g_{im}T_{kl}) -$$

$$-\frac{1}{6}(g_{il}g_{km} - g_{im}g_{kl})T, \tag{5}$$

where V_{iklm}, having the property $g^{il}V_{iklm} = 0$ and which does not vanish in the vacuum around the source, can be identified by the required energy-momentum density tensor for gravitational field. The field equations in the general case are:

$$G_{iklm} = T_{iklm}. \tag{6}$$

The tensors G_{iklm} and T_{iklm} have the symmetry properties as the Riemann tensor and therefore we have 20 equations. The tensor G_{iklm} is a function of the metric tensor g_{ik} which has 6 independent components. The tensor $T_{iklm}^{(m)}$ is combined from the ordinary energy-momentum tensor of the matter T_{ik} and it has 4 independent functions (the energy density ϵ and 3 components of the velocity). These 10 functions are solutions of 10 Einstein equations (6 for independent components of the metric and 4 for independent components of T_{ik}). The new term V_{iklm} has 10 independent components.

So, we have 20 equations for 20 independent functions. If we take solutions of the Einstein equations for some metric and T_{ik}, then we have an additional 10 equations for 10 components of V_{iklm}. This means that the solutions of the Einstein equations exactly define all components of V_{iklm} and we can find V_{iklm} for some standard metric. But if we have some model of the vacuum and calculate V_{iklm} in this model, then we have 10 equations for 10 unknown components of the metric g_{ik} and T_{ik}.

3   The energy-momentum conservation for the system of gravitation and matter

The Riemann tensor can be represented as:

$$R_{iklm} = C_{iklm} + \frac{1}{2}(g_{km}R_{il} - g_{kl}R_{im} + g_{il}R_{km} - g_{im}R_{kl}) - \tag{7}$$

$$-\frac{1}{6}(g_{il}g_{km} - g_{im}g_{kl})R \tag{8}$$

where C_{iklm} is the Weyl tensor with zero 2-index contraction $g^{il}C_{iklm} = 0$. In the vacuum $T_{ik} = T = 0$, $R_{il} = R = 0$, and we have:

$$\frac{1}{\kappa}C_{iklm} = V_{iklm} \tag{9}$$

The covariant derivatives of these 4-index tensors are:

$$
\begin{aligned}
G^i_{.klm;i} &= \frac{1}{\kappa}\left[R^i_{.klm;i} - \frac{1}{6}(g_{km}R_{,l} - g_{kl}R_{,m})\right] = \\
&= T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{,l} - g_{kl}T_{,m}),
\end{aligned} \tag{10}
$$

$$T^{j(m)}_{klm;j} = \frac{1}{2}\left[T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{;l} - g_{kl}T_{;m})\right] = \frac{1}{2}G^i_{.klm;i}. \tag{11}$$

Then we obtain the relationship:

$$V^j_{klm;j} = G^j_{klm;j} - T^{j(m)}_{klm;j} = \frac{1}{2}G^i_{.klm;i}. \tag{12}$$

and, therefore,

$$V^j_{klm;j} = T^{j(m)}_{klm;j} \tag{13}$$

In the vacuum, therefore, there are local conservation laws:

$$G^j_{.klm;j} = V^j_{klm;j} = 0. \tag{14}$$

The integral energy-momentum tensors for matter and gravity can be defined as:

$$P^{(m)}_{ikl} = \int dS_n\sqrt{-g}T^{n(m)}_{.ikl}, \tag{15}$$

$$V_{ikl} = \int dS_n\sqrt{-g}V^n_{.ikl}. \tag{16}$$

In the hypersurface $x^0 = const$ we have:

$$P^{(m)}_{ikl} = \int d^3x\sqrt{-g}T^{0(m)}_{.ikl}, \tag{17}$$

$$V_{ikl} = \int d^3x\sqrt{-g}V^0_{.ikl}. \tag{18}$$

We can construct the angular momentum tensor for the gravitational field as:

$$M^{ilkm} = \int dS_n\sqrt{-g}(x^m V^{nikl} - x^i V^{nmkl}). \tag{19}$$

This 4-index tensor can be interpreted as *the angular momentum tensor* for the gravitational field.

4 Comparison with pseudotensor and Hamiltonian approaches

The pseudotensor t_{ik}, for example, in the Landau-Lifshitz version, has been defined as a part of the Einstein tensor:

$$G^{ik} = \frac{1}{(-g)} \frac{\partial \psi^{ikl}}{\partial x^l} - t^{ik}. \tag{20}$$

The such separation leads to the conservation of the sum of t^{ik} and T^{ik}:

$$\int dS_k (-g)(T^{ik} + t^{ik}) = \int dS_k \frac{\partial \psi^{ikl}}{\partial x^l}, \tag{21}$$

and the right hand term then can be interpreted as a total energy of the system.

Thus, if we want to work with a localizable and tensor form of the gravitational energy-momentum in the vacuum, we can take the Weyl tensor, which has zero 2-index contraction. If we want to work with non-zero 2-index form of the gravitational energy-momentum, we take the pseudotensors or the Hamiltonians, which are non-localizable and non-covariant.

4.1 The geodesic deviation and measurements of the gravitational energy

The equation of geodesic deviation:

$$\frac{D^2 \eta^i}{ds^2} = R^i_{klm} u^k u^l \eta^m \tag{22}$$

we rewrite in vacuum as:

$$\frac{D^2 \eta^i}{ds^2} = C^i_{klm} u^k u^l \eta^m = \kappa V^i_{klm} u^k u^l \eta^m. \tag{23}$$

Therefore, we conclude that the measurements of the geodesic deviations are exactly the measurements of the 4-index energy-momentum tensor of gravitational field V^i_{klm}.

In [2] the energies for standard metrics and some consequences of the proposed treatment of the gravitational energy will be considered.

References

[1] Zakir Z. (1999) *New Equations for the Gravitation with Riemann Tensor and 4-Index Energy-Momentum Tensors for the Gravitation and the Matter.* (gr-qc/9905009), 4 p., in: "Zakir Z. (2003) *Structure of Space-Time and Matter.* CTPA, Tashkent"

[2] Zakir Z. (1999) *Four-Index Equations for the Gravitation and the Gravitational Energy-Momentum Tensor.* (gr-qc/9906039), in: "Zakir Z. (2003) *Structure of Space-Time and Matter.* CTPA, Tashkent"